## Question I

Suppose $A \in M_{n\times n}(\mathbb{C})$ is normal, i.e., $A = QDQ^*$ for some unitary $Q$. Also, we suppose there are $k$ dominant eigenvalues of $A$, i.e.,

$$|\lambda_1| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| \geq \cdots \geq |\lambda_n|$$

Consider the power iteration:

$$\mathbf{x}^{m+1} = A\mathbf{x}^m.$$

Define the angle between a non-zero vector $\mathbf{x}$ and a non-trivial subspace $W$.

$$\begin{aligned}
\cos\angle(\mathbf{x}, W) &:= \max\left\{ \cos\angle(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in W \setminus \{0\}\right\},\\
\sin\angle(\mathbf{x}, W) &:= \min\left\{ \sin\angle(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in W \setminus \{0\}\right\},\\
\tan\angle(\mathbf{x}, W) &:= \frac{\sin\angle(\mathbf{x}, W)}{\cos\angle(\mathbf{x}, W)}.
\end{aligned}$$

Using the given definitions, prove that:

1. Let $W$ be a nontrivial subspace of $\mathbb{C}^n$ and $P$ be the orthogonal projection onto $W$. Then, for $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \notin W$,

   $$\cos\angle(\mathbf{x}, W) = \frac{\|P\mathbf{x}\|}{\|\mathbf{x}\|},\ \sin\angle(\mathbf{x}, W) = \frac{\|\mathbf{x} - P\mathbf{x}\|}{\|\mathbf{x}\|},\ \tan\angle(\mathbf{x}, W) = \frac{\|\mathbf{x} - P\mathbf{x}\|}{\|P\mathbf{x}\|}$$

   Here, $||\cdot||$ is the usual Euclidean norm in $\mathbb{C}^n$.

2. Let $\{\delta_i : i = 1, \cdots, n\}$ be the standard ordered basis for $\mathbb{C}^n$. Let $W_k := \text{span}\{Q\delta_j : j = 1, \cdots, k\}$ and let $\cos\angle(\mathbf{x}^0, W_k) \neq 0$. Then,

   $$\tan\angle(\mathbf{x}^{m+1}, W_k) \leq \frac{|\lambda_{k+1}|}{|\lambda_k|} \tan\angle(\mathbf{x}^m, W_k).$$

## Question II

Consider the linear system $Ax = b$. The (Generalized minimal residual method) GMRES method is a projection method which obtains a solution in the $m$-th Krylov subspace $K_m$ so that the residual is orthogonal to $AK_m$. Let $r_0$ be the initial residual and let $v_0 = r_0$. The Arnoldi process (see the hint below) is applied to build an orthonormal system $v_1, v_2, \ldots, v_{m-1}$ with $v_1 = Av_0/\|Av_0\|_2$ ($\|\cdot\|_2$- the $l^2$ norm). The approximate solution is obtained from the following space

$$K_m = \text{span}\left\{v_0, v_1, \ldots, v_{m-1}\right\}.$$

1. Show that the approximate solution is obtained as the solution of a least-square problem, and that this problem is triangular.

2. Prove that the residual $r_k$ is orthogonal to $\{v_1, v_2, \ldots, v_{k-1}\}$.

3. Find a formula for the residual norm.

4. Derive the complete GMRES algorithm.

*Hint:* The Arnoldi process uses the stabilized Gram-Schmidt process to produce a sequence of orthonormal vectors $v_1, v_2, v_3, \ldots$. Explicitly, the algorithm is as follows:

(1.) Start with an arbitrary vector $v_1$ with norm 1.

(2.) Repeat for $k = 2, 3, \ldots$

   i. $v_k \leftarrow Av_{k-1}$

   ii. for $j$ from 1 to $k-1$

      A. $h_{j,k-1} \leftarrow v_j^* v_k$

      B. $v_k \leftarrow v_k - h_{j,k-1} v_j$

   iii. $h_{k,k-1} \leftarrow \|v_k\|_2$

   iv. $v_k \leftarrow \frac{v_k}{h_{k,k-1}}$